Parametric Uncertainty Analysis of Inverse Linear Electric Circuit Problems

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***Abstract*—This paper focus on the robustness analysis inverse problem of linear electric circuits. The aim is to determine what amount of variations of the input parameters can an electrical or electronic system tolerate, such that the output variables (e.g. conducted noise) does not exceed an upper bound in the frequency domain. Based on the well-known theory of robust control and convex optimization, we propose a procedure to check which commercial values of tolerance can be picked for components of an arbitrary electric circuit, while guaranteeing a prescribed upper bound. The proposed method is applied to an op-amp active band-pass filter and the results are analyzed and discussed.**

Keywords*— *Worst-case Analysis, Uncertainty Analysis, Convex Optimization, Inverse Problem.

I. Introduction

Since the advent of fast and affordable computers, numerical simulations have been utilized widely in analysis and design of electrical and electronic systems. Nowadays, electrical engineers rely heavily on computer simulations to predict the behavior of systems before building a physical prototype. This approach reduces the time-to-market and the production cost, since changing a parameter on a computer model is much more convenient than doing so on a prototype [1]–[3].

Even though software packages available today can carry out computer simulations very accurately, most of them do not take into account the intrinsic parametric uncertainty that exists in almost all applications. Temperature and humidity variation, production dispersion in the fabrication of electrical components, unknown external electromagnetic interference and material properties variability are some examples of the nature of the parametric uncertainty in electric circuits. These phenomena have a significant impact on stability and performance of electrical and electronic systems and therefore it is crucial to take them into account in numerical simulation [4].

There are two major approaches to deal with uncertainty: probabilistic and worst-case. The probabilistic approach considers the input and output variables as random and therefore the methods proposed in this framework are stochastic. In general, it is used either probability density functions or statistical moments to model probabilistic uncertainty. One popular stochastic method for stochastic uncertainty analysis is the Unscented Transform [5], [6]. In the worst-case approach, it is considered that the input and output variables belong to an interval. The goal of the method is to determine an upper bound of a performance, which may be any voltage or current measurement [7], [8].

In the context of numerical simulations, two main problems can be identified: the forward problem and the inverse problem. The forward problem consists of determining the output variable given the values of the input variables. When parametric uncertainty is considered, then the forward problem consists of determining the behavior of the output given the input variables and the uncertainty description, which may be probabilistic or worst-case.

An inverse problem, however, consists of determining the input variables given the values of the output variables. Here, when parametric uncertainty is considered, the inverse problem corresponds to determining the uncertain input parameters given some results on the uncertain output variables. Very few work has been done in the electrical engineering community to tackle the inverse problem with parametric uncertainty, even though the solution of this problem reveals critical information about the system. For instance, consider an electrical circuit in which an upper bound of an arbitrary voltage is defined. In this context, the inverse problem corresponds to computing what quantity of variation can take place on the electrical components such that the aforementioned upper bound is respected.

The uncertainty analysis of inverse problems is challenging. In general, inverse problems have many input variables and few output variables. Since the goal is to compute input variables given the output variables, often the problem is not well-posed. Moreover, this problem is computationally demanding and suffers from the curse of dimensionality. In other words, its computational complexity grows exponentially with the number of unknowns [9].

In this paper, we are going to focus on a specific application of electrical circuits, which is Electromagnetic Compatibility (EMC). In the context of EMC, electrical circuits are used to model low frequency systems up to the low MHz frequency range, such as cables, transmission lines, filters and power converters. There are several EMC standards which must be considered when designing electric and electronic systems. EMC standards are usually upper bounds for the noise generated by electronic systems, in the frequency domain. Undoubtedly, the electrical circuit parameters have an impact on the conducted emissions. For instance, it is known that

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the parasitic resistance of the DC bus capacitor has a strong impact on the noise generated by DC-DC Converters [1].

The problem addressed in this paper consists in determining what amount of variations of the input parameters can an electrical or electronic system tolerate, such that the output variables (e.g. conducted noise) does not exceed an upper bound in the frequency domain. Here, we propose an efficient method to solve the aforementioned problem, based on the Bounded Real Lemma and lineal matrix inequalities (LMIs) [10].

The remainder of the paper is organized as follows: in section II, the problem addressed in this paper is presented. In section III, the methodology based on robust control is explained. Then, in section IV, an application is described. Finally, in sections V and VI, the results are presented and analyzed and a conclusion is given.

II. Problem Statement

Consider an arbitrary electric circuit, composed of linear components such as resistors, inductors, capacitors, voltage sources, current sources, ideal operational amplifiers, voltage measurements and current measurements. Each component value is labeled as θ_k, there are p components and, thus $k = 1, 2, ..., p$. Additionally, each component θ_k can be subjected to a parametric uncertainty δ_k, such that the component value is actually given by $\theta_k = \theta_k^0(1 \pm \delta_k)$, where θ_k^0 is known as the nominal value.

Moreover, this circuit is considered to have one input $U(j\omega)$ and one output $Y(j\omega)$. The input can be an arbitrary voltage or current source and the output can be an arbitrary voltage or current measurement. Let $T_\theta(j\omega)$ be the transfer function of $U(j\omega)$ to $Y(j\omega)$ for each value of θ and let Ω be the set of frequencies ω considered.

Consider also an upper bound in the frequency domain η and define Θ as the region representing the set of all feasibility values of uncertain parameters θ_k, $k = 1, 2, ..., p$. Then, the problem consists on determining the biggest feasibility region in the uncertain parameter space Θ, such that $|T_\theta(j\omega)| < \eta, \forall\omega$ in Ω and $\theta_k \in \Theta$.

In this paper, in particular, we are going to focus on the first stage of this work, which can be seen as the following analysis problem: consider an arbitrary electric circuit, where it is known the set of different commercial values of tolerance for each components θ_k, $k = 1, 2, .., p$ and let η be an given upper bound in the frequency domain. The question is to determine the tolerance limit for each components which ensure that $|T_\theta(j\omega)| < \eta, \ \forall\omega$.

III. Robust Worst-Case Formulation

Note that, the analysis problem is an infinity-dimensional problem, which can be hard to find a solution. In order to obtain a solution, we consider that the electric circuit can be modeled as the following uncertain dynamical system

$$S: \begin{array}{rl} \dot{x}(t) & = A(\theta)x(t) + B(\theta)u(t) \\ y(t) & = C(\theta)x(t) + D(\theta)u(t) \end{array} \tag{1}$$

where $\theta = (\theta_1, \ldots, \theta_p) \in R^p$ is the vector of uncertain parameters, $x \in R^n$ is the state, $u(t) \in R$ is the input signal which belongs to $\mathcal{L}^2$, $y(t) \in R^1$ is the output signal of interest, and $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ are real matrices of appropriate dimensions that depend affinely on the parameter vector θ. Note that the set Θ of uncertain parameters can be seen as convex region called polytope. Moreover, from the theory of optimal and robust control, it is well known that the system H_∞ norm represents the largest possible frequency gain for each given θ and is given by [11]

$$\|T_\theta\|_\infty := \sup_w \sigma_{max}\{T_\theta(j\omega)\} \tag{2}$$

where σ_{max} is the maximum singular value and T_θ is the transfer function from $U(j\omega)$ to $Y(j\omega)$. Thus, the analysis problem can be reformulated as to find the set Θ, such that $\|T_\theta\|_\infty < \eta$ for a given upper bound η and for all $\theta \in \Theta$.

Next lemma presents the Bound Real Lemma, which gives sufficient conditions to ensure that the $||T_\theta||_\infty$ is bounded for a given scalar for all uncertain parameters in a given region [10], [12].

Lemma 1: Consider the system (1) and $\eta > 0$ a given scalar. Let Θ be a given polytope. The system (1) is stable and $||S||_\infty < \eta$, for all $\theta \in \Theta$ if there exists a matrix $P > 0$ such that for all θ in the vertices of Θ:

$$. \quad \begin{bmatrix} A^T(\theta)P + PA(\theta) & PB(\theta) & C^T(\theta) \\ B^T(\theta)P & -\eta I & D^T(\theta) \\ C(\theta) & D(\theta) & -\eta I \end{bmatrix} < 0, \tag{3}$$

□

Since the conditions of Lemma 1 are cast as a set of linear matrix inequalities (LMIs), this is a convex optimization problem which can be solved using known numerical solvers and an optimization toolbox, such as YALMIP [13].

In the next section, we will apply the above procedure in order to determine if the selected values of tolerance for some components assure the maximum gain of an active band-pass filter to be below a selected upper bound.

IV. Procedure

An op-amp active band-pass filter will be considered as an example to show the proposed procedure. The circuit is shown in Fig. 1, where all the resistors are considered to be ideal and both C_1 and C_2 are subjected to parametric uncertainties.

The state-space model for the circuit is shown in equation (4), where the output of the state-space model is chosen to be the output voltage of the circuit.

$$\begin{aligned} \begin{bmatrix} \frac{dv_o(t)}{dt} \\ \frac{dv_x(t)}{dt} \end{bmatrix} &= \begin{bmatrix} \frac{-(C_1+C_2)}{C_1C_2R_2} & \frac{R_1+R_3}{C_1R_1R_3} \\ \frac{-1}{C_2R_2} & 0 \end{bmatrix} \begin{bmatrix} v_o(t) \\ v_x(t) \end{bmatrix} + \begin{bmatrix} \frac{-1}{R_1C_1} \\ 0 \end{bmatrix} v_i(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_o(t) \\ v_x(t) \end{bmatrix} \end{aligned} \tag{4}$$

Since equation (4) presents a non-linear relation between C_1 and C_2, it is not possible to use Lemma 1 directly. Instead, we define $\theta = \{\theta_1 = 1/C_1; \theta_2 = 1/C_2\}$ and thus each system

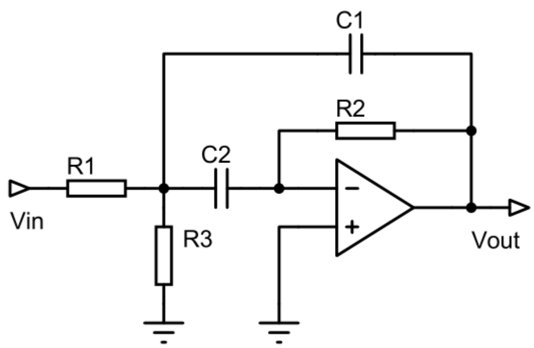

Fig. 1. Active band-pass filter.

matrix can be written as a linear function of θ_1 and θ_2. Hence, the matrices A, B, C and D of the considered system as (1) are given by:

$$A(\theta) = \begin{bmatrix} -\frac{\theta_1}{R_2} - \frac{\theta_2}{R_2} & \frac{\theta_1(R_1+R_3)}{R_1R_3} \\ -\frac{\theta_2}{R_2} & 0 \end{bmatrix}$$
$$B(\theta) = \begin{bmatrix} -\frac{\theta_1}{R_1} \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$D = \begin{bmatrix} 0 \end{bmatrix}$$

Now, the LMI constraint in (3) is evaluated considering the tolerances δ_1 and δ_2, which will result in four vertices given by the pairs (θ_1^i, θ_2^j) $i, j = 1, 2.$, where $\theta_1^1 = 1/C_1(1+\delta_1)$, $\theta_2^1 = 1/C_2(1+\delta_2)$, $\theta_1^2 = 1/C_1(1-\delta_1)$ and $\theta_2^2 = 1/C_2(1-\delta_2)$. The upper bound η considered will be slightly superior to the nominal case.

The values considered for δ_1 and δ_2 are the typical tolerances for ceramic capacitors which are 1%, 2%, 5%, 10%, 20% and 80%. The problem will be solved for each pair of values of δ_1 and δ_2, including the nominal case. By solving in this fashion, each time that the problem is feasible, we guarantee that the voltage gain of the circuit is under the upper bound η. This is true for any value of C_1 and C_2 inside the range of the selected tolerances (δ_1,δ_2) and thus we can use that pair of commercial tolerances to get the expected response.

V. Numerical Results

The proposed method is going to be tested considering $R_1^0 = R_2^0 = R_3^0 = 1k\Omega$ and the following nominal values of capacitance $C_1^0 = C_2^0 = 10\mu F$. When both capacitors are not uncertain, that is both are fixed to be $10\mu F$, the maximum voltage gain of the frequency response of the circuit is 0.5. Thus, in order to get an upper bound 10% higher than the expected gain for the nominal values, η was set to be 0.55 for this scenario.

Table I shows the results generated for each case, where the number 1 indicates feasibility of Lemma 1 and the number 0 indicates unfeasibility. By these results one can see that if only one parameters is uncertain, that is, if only one capacitor is allowed to be uncertain and the other is assumed to be constant, the proposed approach get solution in 71% of the cases. On the other hand, in the case where at least one capacitor's tolerance is 20% or 80%, it is not possible to ensure that the voltage gain will be less that 0.55.

Figure 2 helps to visualize part of the problem that we are solving, since it shows the bode diagram for the voltage gain of the four vertices for each one of the tolerance pairs. If only one of the responses of a tolerance pair is above the value of η, the LMI problem should be unfeasible and the response of the circuit can not be guaranteed.

In Figure 3, a zoom of the bode plots previously seen for five particular tolerance pairs is shown: the nominal case, two feasible cases and two unfeasible cases. By comparing the plot to the results of Table I, it can be seen that for the unfeasible cases at least one of the frequency responses corresponding to a vertex intersects the upper bound. This fact is expected from the formulation of the problem.

Finally, to further check the results, the frequency response of six particular pairs of interior values for C_1 and C_2 is plotted, where the first three pairs are guaranteed by our method to have a maximum gain below η and the three following pairs we can not guarantee the response to be under that same upper bound. The results found are shown in the Figure 4, where it can be seen that the response for the first three pairs are under the upper bound as expected, and that two of the three responses for the last pairs have maximum gains above the value of η. It should be noted that, for the last pair, even though the response is below η, the proposed method can not guarantee that result because it considers all the possible values for a tolerance pair in the region defined by the vertices of the uncertain parameters.

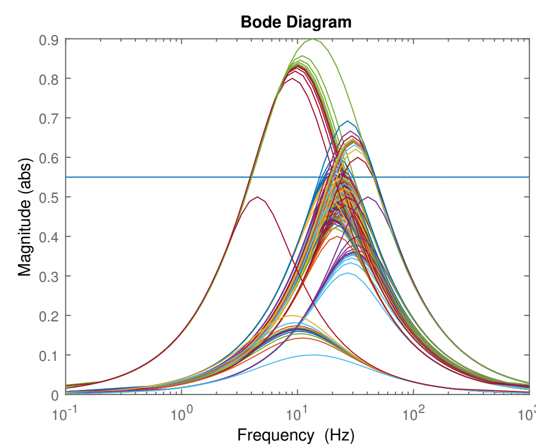

Fig. 2. Frequency response for all tolerance pairs of capacitance.

VI. Conclusion

This paper proposed a procedure that, given a linear electrical circuit state-space model and an upper bound of any

TABLE I
PROBLEM FEASIBILITY FOR TOLERANCE RANGE OF CAPACITANCE

δ_2 \ δ_1	0%	1%	2%	5%	10%	20%	80%
0%	1	1	1	1	1	0	0
1%	1	1	1	1	0	0	0
2%	1	1	1	1	0	0	0
5%	1	1	1	1	0	0	0
10%	1	0	0	0	0	0	0
20%	0	0	0	0	0	0	0
80%	0	0	0	0	0	0	0

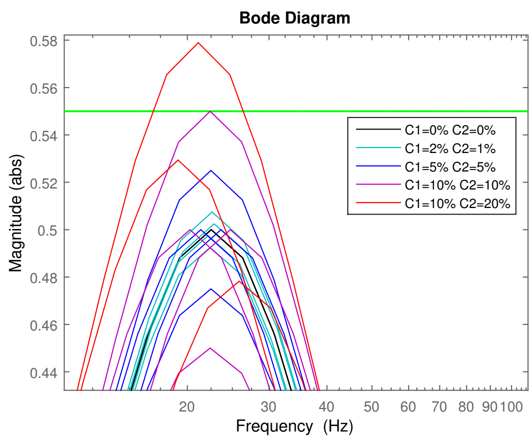

Fig. 3. Frequency response for some tolerance pairs.

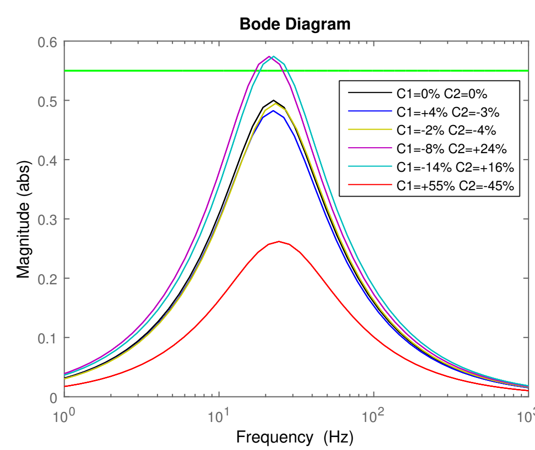

Fig. 4. Frequency response for particular values of capacitance.

transfer function between electrical sources and measurements, verifies if the system's response is always less than the bound, even in the presence of parametric uncertainty of the electrical components.

The analysis problem was formulated as classical H_∞ robust analysis problem, which can be solved using the Bound Real Lemma and convex optimization solvers.

The procedure was applied to a band-pass filter, with two uncertain capacitors. Five sets of variations were considered for each parameter and the results were obtained and analyzed. It was shown that the method performed as expected and that the variation of capacitance values due to tolerance must be taken into account when designing analog filters.

The authors intend, in future work, to generalize the proposed method to N input variables and M output variables. Also, another perspective is to solve the general problem, which corresponds to finding the entire feasibility set Θ iteratively.

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